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Solution of a nonlinear boundary value problem for the integro-differential Fredholm equation by the parameterization method

ABSTRACT

of the PhD thesis for the degree of Doctor of Philosophy (PhD)
in the specialty «6D060100-Mathematics»

The relevance of the study is twofold: on one side, it is conditioned by many applications of integro-differential equations in solving problems in natural science; on the other side, there is the need to develop new constructive methods that allow efficiently determine the solvability of nonlinear problems for integro-differential equations and find their solutions.

At the beginning of the last century Volterra showed that the problem of equilibrium of an elastic solid taking into account the after-effect phenomenon leads to an integro-differential equation. The emergence of integro-differential equations in the study of various trends in physics, chemistry, biology, economics and others had a great impact on their development and wide application. Boundary value problems for integro-differential equations have been studied by many scientists; various solution methods and technics of finding them have been proposed.

The purpose of the dissertation is to develop effective methods for finding solutions and applying the Dzhumabaev parametrization method in solving initial and boundary value problems for nonlinear Fredholm integro-differential equations with a nonlinear integral part.

The following **research objectives** are stated in the dissertation:

a) Establishment of the conditions for the existence and uniqueness of a solution of the special Cauchy problem for systems of nonlinear integro-differential equations with a nonlinear integral part;

b) Construction of a new general solution of the Fredholm integro-differential equation with the nonlinear integral part and studying its properties;

c) Obtaining the solvability conditions of a nonlinear boundary value problem for the Fredholm integro-differential equation;

d) Construction of a general solution for the quasilinear Fredholm integro-differential equation and finding a solution to the boundary value problem;

e) Establishing the solvability condition for the Cauchy problem and the boundary value problem for the Fredholm integro-differential equation by the averaging method.

The object of the study is the initial and boundary value problems for nonlinear integro-differential Fredholm equations with a nonlinear integral part.

The subject of the study is the solvability of boundary value problems for Fredholm integro-differential equations with a nonlinear integral part, the justification of the averaging method for initial and boundary value problems with a small numerical parameter.

Scientific novelty.

1. The Dzhumabaev parametrization method is applied to the Fredholm integro-differential equation with a nonlinear integral part.

2. The conditions for solvability of the special Cauchy problem for the nonlinear integro-differential Fredholm equation are obtained.

3. A new general solution of the Fredholm integro-differential equations with a nonlinear integral part is constructed.

4. The Dzhumabaev parametrization method is applied to the boundary value problem for nonlinear integro-differential Fredholm equations.

5. The initial and boundary value problems for the nonlinear integro-differential Fredholm equations were solved by the averaging method.

Theoretical and practical significance of the results. The results are mainly theoretical. The scientific significance of the work consists in the development of a constructive method of research and solving problems for integro-differential equations with nonlinear integral part.

Publications. 10 papers have been published on the subject of the dissertation, including 3 publications in the ranking scientific journals, indexed in the database of Scopus, 3 publications in scientific publications, included in the list recommended by the Committee for Quality Assurance in the Field of Science and Higher Education of the Ministry of Science and Higher Education of the RK for publication of the main scientific results of scientific activity, 4 publications in the materials of international conferences, including 1 publication in the proceedings of foreign conferences.

– 3 publications in journals included in the list recommended by the Committee for Quality Assurance in the Field of Science and Higher Education of the Ministry of Science and Higher Education of the Republic of Kazakhstan:

1. Dzhumabaev D.S., **Karakenova S.G.** An iterative method for solving the special Cauchy problem for the system of nonlinear integral part // Kazakh Mathematical Journal. 19(2) 2019,pp. 49-58.

2. Kadirbayeva Zh.M, **Karakenova S.G.** Numerical solution of multi-point boundary value problems for essentially loaded ordinary differential equations// Kazakh Mathematical Journal. 20(4) 2020, pp. 47–57.

3. Stanzhitskii A.N., **Karakenova S.**, Zhumatov S.S. On a comparison theorem for stochastic integro-functional equations of neutral type// Journal of Mathematics, Mechanics and Computer Science. Al-Farabi Kazakh National university. №1 (105), 2020, pp 30-45 (WoS, IF(2022)=0.1; JCI(2022) – Q4)

- 3 articles in scientific journals included in the respective third quartile according to the Clarivate Analytics Journal Citation Reports and/or have percentages respectively 33, 35, 68 CiteScore in the Scopus database:

1.Stanzhitskii A.N., **Karakenova S.G.**, Uteshova R.E. Averaging Method and Boundary Value Problems for Systems of Fredholm Integro-Differential Equations// Nonlinear Dynamics And Systems Theory. 20 (1) 2021, pp. 100-113. (Scopus, 38% in Category Mathematical Physics)

2. Assanova A.T., Zhumatov S.S., Mynbayeva S.T., **Karakenova S.G.** On solvability of boundary value problem for a nonlinear Fredholm integro-differential equation // Bulletin of the Karaganda University. Mathematics Series. – 2022. – Vol. №1(105)/2022. – p. 25–34. DOI 10.31489/2022M1/25-34 (WoS, IF(2022)=0.6, JCI(2022) -Q3; Scopus, 35% in Category General Mathematics)

3. Assanova A.T., Mynbayeva S.T., **Karakenova S.G.**, Uteshova R.E. A solution to a nonlinear Fredholm integro-differential equation // Quaestiones Mathematicae. Published online: 09 Mar 2023. https://doi.org/10.2989/16073606.2023.2183157 (WoS, Q3, IF(2022)=0.7, JCI(2022) –Q2; Scopus, 68% in Category Mathematics (miscellaneous)

The main results of the thesis were reported and discussed at international and foreign scientific conferences and seminars:

1. **Karakenova S.** On the solution of the special Cauchy problem for the system of nonlinear Fredholm integro-differential equations// Annual Internat. april mathem. conf. in honor of the Day of Science Workers of the Republic of Kazakhstan. – Almaty: Institute of mathematics and mathematical modeling. 3-5 April 2019, P.97-98.

2. **Karakenova S.G.** On solution to the special Cauchy problem for Fredholm integro-differential equations with nonlinear integral part// International Conference of Young Mathematician. 6-8 June 2019, p.30.

3. **Karakenova S.** Approximate method for solving special Cauchy problem for nonlinear integro-differential equation// Annual internat. april mathem. conf. in honor of the Day of Science Workers of the Republic of Kazakhstan. – Almaty: Institute of mathematics and mathematical modeling. 3-5 April 2022г., P.121-122.

4. Mynbayeva S., **Karakenova S.** On one approach to general solution to a nonlinear Fredholm integro-differential equation // Annual internat. april mathem. conf. in honor of the Day of Science Workers of the Republic of Kazakhstan. – Almaty: Institute of mathematics and mathematical modeling. 4-8 April 2022, P. 149-150.

Approbation of the work. The main results were reported and discussed at the following scientific events:

1. The traditional International Scientific April Conference. Institute of mathematics and mathematical modeling. Almaty, (3-5 April 2019, 3-5 April 2020, 5-8 April 2022);

2. Scientific seminar of the Department of Differential Equations. Al-Farabi Kazakh National University. Almaty, (February 2019г., April 2020);

3. International Conference of Young Mathematicians. Institute of Mathematics of NAS of Ukraine. Ukraine, Kyiv, (June 6-8, 2019);

4. Seminar on Stochastic differential Equations of Taras Shevchenko National University of Kyiv (June 2019).

The structure and scope of the thesis. The dissertation consists of an introduction, three sections (the first section includes 3 subsections, the second section – 3 subsections, the third section – 2 subsections), a conclusion, a list of 132 used sources.

The introduction includes an assessment of the current state of the problems under consideration, provides justification for the need for research work. The introduction describes the relevance and novelty of the topic, the main goals and objectives of the study, the provisions proposed for defence.

The thesis investigates the boundary value problem for the Fredholm integro-differential equation with a nonlinear integral part:

$$\frac{dx}{dt} = A(t)x + \sum_{k=1}^{m} \varphi_k(t) \int_0^T \psi_k(\tau) f_k\big(\tau, x(\tau)\big) d\tau \,, t \in [0,T], x \in R^n, \quad (1)$$

$$g[x(0), x(T)] = 0. \quad (2)$$

The first section discusses the general scheme of the Dzhumabaev parametrization method for the Fredholm integro-differential equation with a nonlinear integral part. The main idea of the parametrization method is as follows: the interval $[0,T]$ is divided into N parts by points $t_0 = 0 < t_1 < \cdots < t_N = T$ and the partition is denoted by Δ_N. We then obtain a system of nonlinear integro-differential equations on the partition subintervals. Then we introduce the additional parameters $\lambda_r \mathrel{\widehat{=}} x_r(t_{r-1})$ and the substitutions $u_r(t) = x_r(t) - \lambda_r,\ t \in [t_{r-1}, t_r), r = \overline{1,N}$. On the partition subintervals we obtain the system of nonlinear integro-differential equations with parameters:

$$\frac{du_r}{dt} = A(t)(u_r + \lambda_r)$$

$$+ \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \psi_k(\tau) f_k\big(\tau, u_j(\tau) + \lambda_j\big) d\tau \,, t \in [t_{r-1}, t_r), r = \overline{1,N} \quad (3)$$

and initial conditions

$$u_r(t_{r-1}) = 0, \qquad r = \overline{1,N}\,. \tag{4}$$

Problem (3), (4) is called the special Cauchy problem for the system of nonlinear integro-differential equations with parameters.

This special Cauchy problem is the main auxiliary problem in solving the boundary value problem for the nonlinear Fredholm integro-differential equation with a nonlinear integral part. Solvability conditions are established and methods of solving problem (3), (4) are proposed.

The iterative method with damping factors for solving the special Cauchy problem for the system of nonlinear integro-differential equations with parameters (3), (4) is justified, and some examples describing the iterative process are given.

In the second section, the boundary value problem for systems of nonlinear integro-differential Fredholm equations with a nonlinear integral part is considered. The Δ_N −general solution of the Fredholm integro-differential equation with a nonlinear integral part is constructed and its properties are identified. Substituting the Δ_N -general solution into the boundary condition and continuity conditions, we obtain a system of nonlinear algebraic equations. In accordance with the conditions of the theorem put on the selected partition Δ_N, the solvability of the boundary value problem was established in terms of the solvability of the system of nonlinear algebraic equations.

In addition, the quasilinear integro-differential Fredholm equation corresponding to equation (1) is considered. Based on the solution of the linear part of this equation, the Δ_N −general solution of the quasilinear integro-differential Fredholm equation was constructed and applied to the study of the boundary value problem. The conditions for the solvability of the linear boundary value problem for the quasi-linear integro-differential Fredholm equation are established.

In the third section, the averaging method is applied to a boundary value problem for the nonlinear Fredholm integro-differential equation with a small parameter. This problem is reduced to a boundary value problem for nonlinear ordinary differential equations. The boundary value problem for the nonlinear integro-differential Fredholm equation is investigated. The solvability conditions of the problem under consideration are established in terms of the solvability of the boundary value problem for ordinary differential equations.